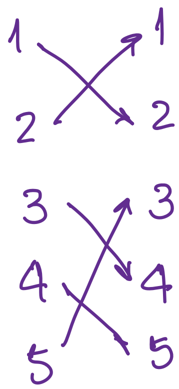
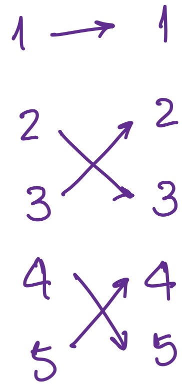
Randomized routing:

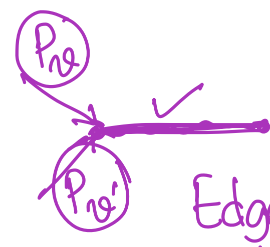
- Let $G=(V,E)$ be a graph s.t each node represents a computing node and edges are communication links/channels.
- Each node $v \in V$ wants to send a packet P_v from itself to a node $\pi(v)$ where π is a permutation.

Ex:





Permutation routing.



Edges are equip w/ Buffers/Queues.

- Constraints: Packets/Messages can only traverse the edges and each edge can carry exactly 1 packet/message at a time.

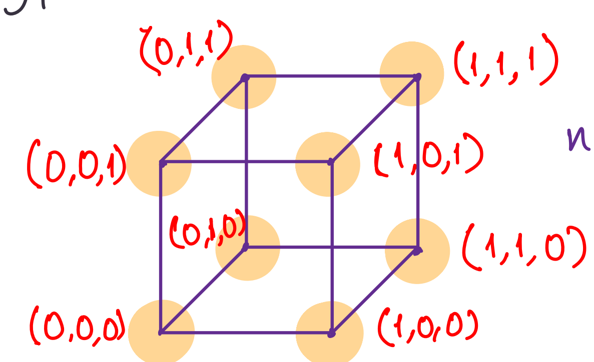
Want: A routing scheme to pass the messages to the intended destinations under the said constraints s.t the total time for all messages to reach their dest. is minimized.

Let G for our study be a hypercube on n bits.

$V = \{0,1\}^n \qquad |V| = 2^n$

$E = \{(u, u \oplus e_i) \mid u \in V \text{ and } i \in [n]\}$

$|E| = \frac{2^n \cdot n}{2}$



$n=3$

$\pi(u) \xleftarrow{P_u} u, \quad v \xrightarrow{P_v} \pi(v)$.

- We will only consider Oblivious routing schemes. That is, the routing scheme of a packet P_v from $v \to \pi(v)$ is independent of the targets of other packets.

"Oblivious Permutation Routing Schemes".

Bit-fixing scheme.

- Flip the individual bits in order to go from $v \to \pi(v)$. (say from MSB to LSB).

Ex: $v = 10110$ to $\pi(v) = 00101$.

$\underline{1}0110 \to 001\underline{1}0 \to 0010\underline{0} \to 00101$

Remark: If more than one packet wants to traverse an edge at the same time, break ties "arbitrarily" and add the rest to a FIFO queue that is associated w/ each edge.

Deterministic routing via bit-fixing can be bad.

Lemma: There are permutations $\pi : \{0,1\}^n \to \{0,1\}^n$ for which bit-fixing scheme requires at least $\frac{2^{n/2}}{n}$ steps to transfer all the messages. $(1,1,0,0) \to (0,0,1,1)$

$(0,1,0,0) \to (0,0,0,0) \to (0,0,1,0)$

Proof: Consider the permutation that maps $(\underline{x}, \underline{0})$ (where $\underline{x}$ is $\frac{n}{2}$ length bit vector and $\underline{0}$ is $\frac{n}{2}$-length zero vector) to $(\underline{0}, \underline{x})$. That is $\pi((\underline{x}, \underline{0})) = (\underline{0}, \underline{x})$. Because of the bit fixing scheme this packet has to be routed through $(\underline{0}, \underline{0})$.

There are $2^{n/2}$ many choices for x and thus $2^{n/2}$ packets will be routed through $(\underline{0},\underline{0})$. Since the total no. of outgoing edges are n at each vertex; the total time of sending all the $2^{n/2}$ packets through the n edges is at least $\frac{2^{n/2}}{n}$. ∎

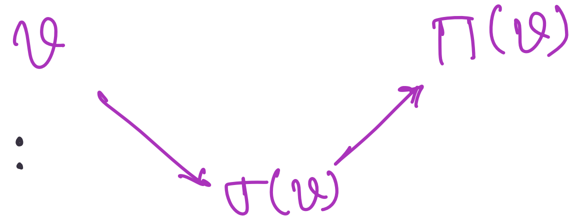
Further the following is known.

<u>Theorem</u>: For any deterministic oblivious routing protocol, there exists a permutation $\pi : \{0,1\}^n \to \{0,1\}^n$ which requires at least $\sqrt{\frac{2^n}{n}}$ steps.

Randomized routing is better. [Valiant]

We shall implement bit fixing scheme in 2 steps with randomness.

For each $v \in \{0,1\}^n$, given π :



Randomness: Pick an "intermediate stop" uniformly and randomly. Let us call this $\sigma(v)$.

Step 1: Route the packet P_v from $v \to \sigma(v)$ using bit fixing scheme

Step 2: Route the packet from $\sigma(v) \to \pi(v)$ using bit fixing scheme.

Theorem: With a high probability all packets will be routed to their destinations in at most $14n$ steps.

The proof of this theorem uses 3 parts as follows.

Claim 1: If $v \to u$ is a path in a bit fixing scheme and $v' \to u'$ is another path in this bit fixing scheme, then these paths do not intersect again once they separate.

$S = \{v' \mid P(v) \cap P(v') \text{ intersect}\}$

Lemma 2: Fix any det. routing scheme. Assume that path from $v \to \pi(v)$ is $e_1, \dots, e_k$. Let S be the set of other vertices v' s.t path from $v' \to \pi(v')$ traverses one of the edges $e_1, \dots, e_k$. Then packet sent from v to $\pi(v)$ takes at most $\underline{k + |S|}$ steps to reach its destination.

Lemma 3: Let $\sigma(v)$ be chosen uniformly and independently for each $v \in \{0,1\}^n$. Let $P(v)$ be the path between v and $\sigma(v)$ obtained by bit fixing scheme. Then with a prob of at least $1 - 2^{-n}$, any path $P(v)$ intersects with at most $6n$ paths $P(w)$ for $w \neq v$.

$|S'| \leq 6n$ w.p. $1-2^{-n}$

Lemma 3': $\sigma(v) \to \pi(v)$. / $S' = \{w \mid \sigma(w) \rightsquigarrow \pi(w) \cap \sigma(v) \rightsquigarrow \pi(v)\}$

Assuming Claim 1, Lemmas 2 and 3, we can infer that step 1 takes at most $n + 6n$ steps w.p $(1 - 2^{-n})$ and same for step 2 using similar arguments. So, with a union bound, we get that with prob $1 - 2^{-(n-1)}$, all packets are routed in at most $14n$ steps.

Lemma 3'.

Proof of Lemma 3:

Estimate # of paths that intersect.

$\leq \sum_{e \in P(v)}$ Esti of paths going through edge e.

For $w, v \in \{0,1\}^n$, let $X_{w,v}$ be an indicator r.v for the event that $P(v)$ and $P(w)$ intersect. We want to estimate $X_v = \sum_{w,v \in \{0,1\}^n} X_{w,v}$.

v $e_1, \ldots, e_k$ $\sigma(v)$

$Y_{e_1, w}$

$\leq \sum_{e \in P(v)} \sum_{w} Y_{e,w}$

w

For an edge $e = (u, u \oplus e_a)$ and a vertex $w \in \{0,1\}^n$, let $Y_{e,w}$ be an indicator r.v for the event that $P(w)$ passes through e. Thus no. of paths that pass through a particular edge e is given by $\sum_{v \in V} Y_{e,v}$.

From bit fixing scheme, $P(v)$ passes through $e = (u, u \oplus e_a)$ iff

$$v_i = u_i \quad \forall \; i > a$$
$$\sigma(v_i) = u_i \quad \forall \; i < a$$

$v_1, v_2, \ldots, v_{a-1}, v_a, v_{a+1}, \ldots, v_n$

$u_1, u_2, \ldots, u_{a-1}, u_a, u_{a+1}, \ldots, u_n$

$\sigma(v)_1, \sigma(v)_2, \ldots, \sigma(v)_{a-1}, \sigma(v)_a, \sigma(v)_{a+1}, \ldots, \sigma(v)_n$.

Let $A_e = \{ v \in \{0,1\}^n \mid u_i = v_i \;\; \forall \; i > a \}$. A path P_v passes through e with a non-zero prob only if $v \in A_e$.

For any $v \in A_e$,

Fixed edge e, v

$$\Pr[Y_{e,v} = 1] = \Pr[\sigma(v)_i = u_i \;\; \forall \; i < a]$$
$$= \frac{1}{2^{(a-1)}}$$

// Prob that $\sigma(v)$ has the same first $(a-1)$ bits as u.

For any edge, the expected no. of paths going through an edge e is

$$\mathbb{E}\left[\sum_{v \in V} Y_{e,v}\right] = \sum_{v \in V} \mathbb{E}[Y_{e,v}]$$

Claim:

$|A_e| = 2^a$

$$= \sum_{v \in V} \Pr[Y_{e,v} = 1]$$

$$= \sum_{v \in A_e} \Pr[Y_{e,v} = 1]$$

$$= 2^a \cdot 2^{-(a-1)} = 2.$$

Two paths intersect if some edge $e \in P(v)$ and $P(w)$.

$$X_{v,w} \leq \sum_{e \in P(v)} Y_{e,w}$$

$X_{v,w} = 1$ if $\sum_{e \in P(v)} Y_{e,w} \geq 1$.
0 o/w.

Total no. of paths that intersect v. is

$X_v = \sum_w X_{v,w}$ and thus,

$$\mathbb{E}[X_v] = \sum_w \mathbb{E}[X_{v,w}]$$

$$= \sum_w \Pr[X_{v,w} = 1]$$

$$= \sum_w \Pr\left[\sum_{e \in P(v)} Y_{e,w} \geq 1\right]$$

$$\leq \sum_w \mathbb{E}\left[\sum_{e \in P(v)} Y_{e,w}\right] \quad // \text{Markov}$$

$$= \sum_{e \in P_v} \mathbb{E}\left[\sum_{w} Y_{e,w}\right]$$ // interchanging summations.

$$\leq n \cdot 2$$ // at most n edges in $P(v)$.

Now, using Chernoff, we can show that

$$\Pr[X_v \geq 6n] \leq 2^{-2n}.$$

$$\Pr[X > (1+\overset{\delta=2}{\delta})\underset{\leq 2n}{\mu}] \leq \exp\left(-\frac{\delta^2}{\delta+2} \cdot \mu\right)$$

and

$$\Pr[\exists v \text{ s.t } X_v \geq 6n] \leq 2^n \cdot 2^{-2n} = 2^{-n}.$$

So with a prob of at least $(1-2^{-n})$ all routing in step 1 happens in at most $6n$ steps.

Proof of Claim 1:

Consider the paths $v \rightsquigarrow u$ given by $w_1, w_2, \dots, w_m$ and $v' \rightsquigarrow u'$ given by $w'_1, \dots, w'_{m'}$. Assume that $w_i = w'_j$ and $w_{i+1} \neq w'_{j+1}$; that is paths separated.

Let $\underbrace{w_{i+1} = w_i \oplus e_a}$ and $\underbrace{w'_{j+1} = w'_j \oplus e_b}$. (W.L.O.G $a < b$).

$$w_{\ell,a} = w_{i,a} \oplus 1 \quad \forall \ell \geq i+1 \qquad\qquad w'_{\ell,a} = w'_{j,a} = w_{i,a} \quad \forall \ell \geq j.$$

Thus after the separation they can never intersect again.

Proof of Lemma 2:

wait time $\le |S|$

$S = \{ v' \mid$ paths of v and v' intersect $\}$.

Suppose P_v and $P_{v'}$ reach an edge e and packet v' is given a priority. Then P_v generates a token and places it on $P_{v'}$. If $P_{v'}$ encounters $P_{v''}$ and $P_{v''}$ is given priority that token is transferred to $P_{v''}$ (where $v'' \in S$).

Observe that wait-time $=$ # of tokens generated.

$v \longrightarrow \pi(v)$ be given by edges $e_1, \dots, e_k$.

Let $P_{v'}$ get a token and traverse $e_i, \dots, e_j$ $(j > i)$ and takes an edge other than e_{j+1} afterwards. Then this token goes away from $P(v)$ and can never come back (due to Claim 1).

Observe that 2 tokens can never be on the same packet $P_{v'}$. So total no. of tokens $\le |S|$.

$\Rightarrow$ Total time taken $\le \underline{k + |S|}$.