(CS1.406) Advanced Algorithms (Spring 2022) Due: 12.01.2022

Problem Set 1

Instructions:

- Discussions amongst the students are not discouraged, but all writeups must be done individually and must include names of all collaborators.
- Referring sources other than the lecture notes is discouraged as solutions to some of the problems can be found easily via a web search. But if you do use an outside source (eg., text books, other lecture notes, any material available online), do mention the same in your writeup. This will not affect your grades. However dishonesty of any sort when caught shall be heavily penalized.
- Be clear in your arguments. Vague arguments shall not be given full credit.

---

1. Suppose you are given a two-sided[1] coin for which the probability of HEADS (say p) is unknown. How can you use this coin to generate unbiased[2] coin flips (give a scheme)? [5 marks]
2. We are given a two-sided and unbiased coin and another coin which has HEADS on both its sides. We choose to pick one of the two given coins randomly with equal probability and flip it. Given that the flip was heads, what is the probability that I flipped the two-headed coin? [6 marks]
3. Let the events $E_1, E_2, \ldots, E_n$ be mutually independent. Then show that the events $\overline{E}_1, \overline{E}_2, \ldots, \overline{E}_n$ are also mutually independent. [5 marks]
4. Consider a set of integers $\{1, 2, \ldots, n\}$ (denoted by $[n]$). We generate a subset X of $[n]$ using the following random process – a two-sided and unbiased coin is flipped independently for each element a of the set $[n]$ and we add a to the set X if and only if the coin lands HEADS.
   (a) What is the probability distribution over all subsets of $[n]$ under this process.
   (b) Suppose two sets X and Y are chosen independently and uniformly at random from all the subsets of $[n]$, then determine
      i. the probability that X is a subset of Y, and
      ii. the probability that $X \cup Y = \{1, 2, \ldots, n\}$.

   [2+6 marks]
5. There are 22 students in a class and a bowl contains 22 envelopes which are identical on their exterior but each envelope contains a chit with a different student's name. Each student picks a random envelope from the bowl. What is the expected number of students who pick an envelope containing their own name. [5 marks]
6. Suppose you pick a graph on n vertices by picking each edge with probability p independently of the others. What is the expected number of triangles in the graph. [6 marks]

[1] A coin is said to be two-sided if it has HEADS on one side and TAILS on the other

[2] A coin is said to be unbiased if it has an equal probability of HEADS and TAILS.