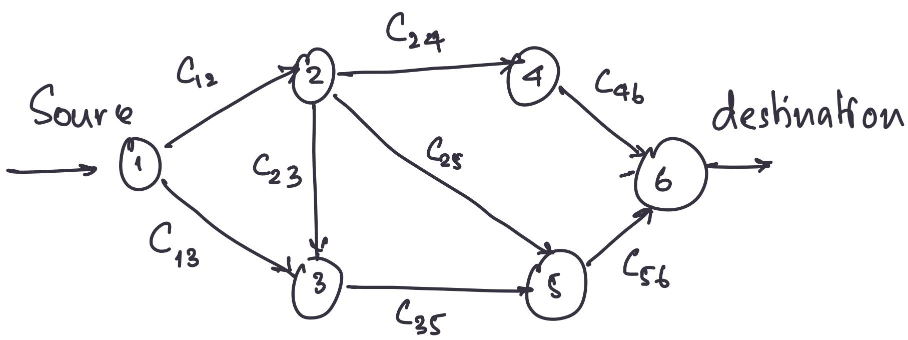
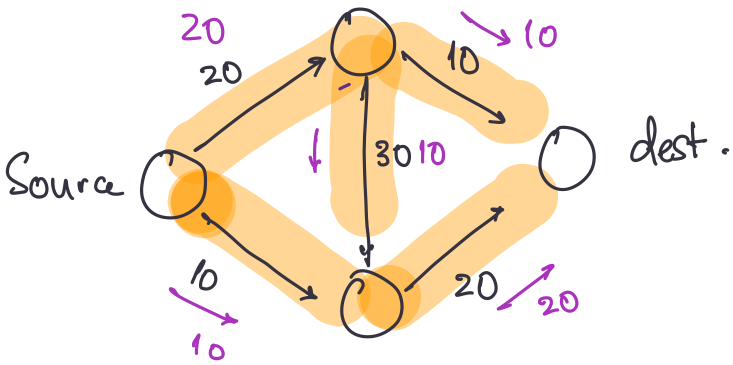
Network Flows.





Conservation of flow:

Any flow that reaches the node is exactly the flow that leaves it.

Flow problem: Given inputs

→ Graph $G = (V, E)$ and for every edge we have a capacity defined.

Properties of capacity: $c(u \to v) \geq 0 \quad \forall \; u \to v \in E$

and any flow $f(u \to v)$ must be s.t

$$0 \leq f(u \to v) \leq c(u \to v).$$

Cut $(S, \underbrace{V \setminus S}_{T})$:

Let S contain source and T contain dest.

$$\text{Capacity}(S, T) = \sum_{u \in S} \sum_{v \in T} c(u \to v)$$

$$\text{Min cut}(G) = \min_{S \subseteq V} \text{capacity}(S, T).$$

1. For every edge $u \to v \in E$, we can define $\underbrace{f(u \to v)}_{\text{flow along the edge } u \to v}$.

$0 \le f(u \to v) \le c(u \to v)$ $\qquad$ $f(u \to v) = 0$ if $u \to v \notin E$.

constraints over edges

2. For any vertex $v \in G$,

$$\sum_w f(w \to v) = \sum_u f(v \to u)$$

constraints over nodes.

$\to$ flow/Throughput $\to$ $\underline{\sum_w f(s \to w)}$ subject to 1 and 2

// maximize

$\sum_u f(u \to t)$

Any flow that constraints 1 and 2 is called "feasible".

<u>Thm:</u> If f is any feasible flow and (S,T) is any (source, dest)-cut, then total flow $|f|$ is at most the capacity of the cut.

$|f| = \sum_w f(s \to w)$

$\sum_w f(s \to w) - \sum_u \cancel{f(u \to s)}^{\,0}$

<u>Pf:</u> $|f| = \sum_{w \in V} f(s \to w) = \sum_{w \in V} f(s \to w) + \underbrace{\left(\sum_{v \in S \setminus \{s\}} \left(\sum_w f(v \to w) - \sum_u f(u \to v) \right) \right)}_{0}$

$$= \sum_{v \in S} \sum_w f(v \to w) - \sum_{v \in S} \sum_u f(u \to v)$$

$$= \sum_{v \in S'} \sum_{w \notin S'} f(v \to w) - \sum_{v \in S'} \sum_{u \notin S'} f(u \to v)$$

$$= \sum_{v \in S'} \sum_{w \in T} f(v \to w) - \underbrace{\sum_{v \in S} \sum_{u \in T} f(u \to v)}_{\geq 0}.$$

$A - B \leq A$ if $B \geq 0$

$$\leq \sum_{v \in S} \sum_{w \in T} f(v \to w)$$

$$\leq \sum_{v \in S} \sum_{w \in T} c(v \to w) \quad \leftarrow \text{Constraint 1.}$$

$$= \text{Capacity}(S,T) \quad \leftarrow \text{defn.}$$

$$\underbrace{\text{any feasible Flow} \leq \text{capacity of any cut}}$$

maximum flow $= \max \{$flow value over all feasible flows$\}$

min-cut $= \min_{S,T} \{$capacity of the S,T cut$\}$.

max-flow $\leq$ min cut.

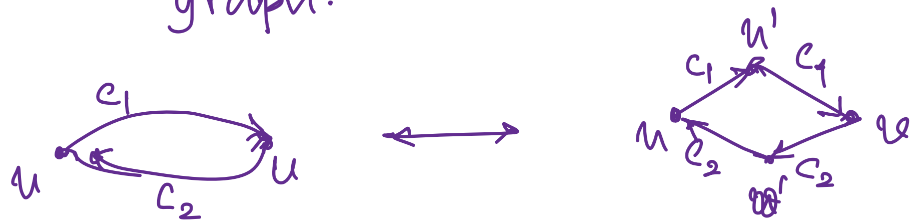
Also, we can show that max flow $\geq$ min cut.

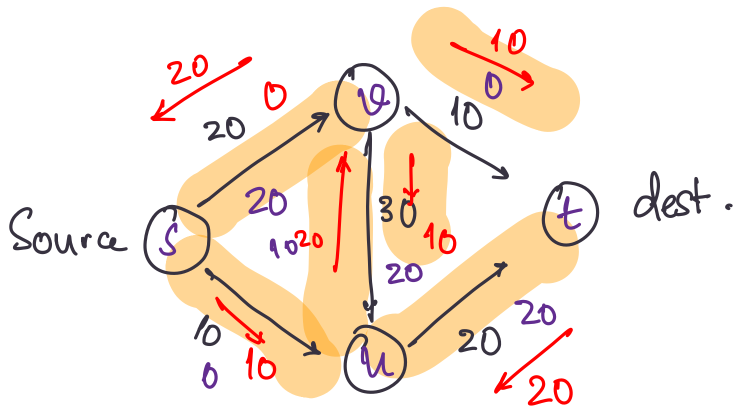
max flow = min cut.

Residual capacity:

$$c_f(u\to v) = \begin{cases} c(u\to v) - f(u\to v) & \text{if } u\to v \in E \\ f(v\to u) & \text{if } v\to u \in E \\ 0 & \text{o/w.} \end{cases}$$

Assumption: No parallel edges in the underlying dir. graph.





$c_f(s,v) = 20-20=0$

$c_f(v,u) = 30-20=10$

$c_f(s,u) = 10-0=10$

$c_f(u,t) = 0$

$c_f(v,t) = 10$

$c_f(u\to v) = f(v\to u) = 20$

$v\to u \in E$