Network Flows (contd.)

$$\max_{\Pi : s \rightsquigarrow t \text{ paths}} \left\{ \min_{e \in \Pi} c(e) \right\}.$$

G_f : Residual graph after f.

Let Δ be a threshold.

Let $G_f(\Delta)$ be the graph obtained by deleting all edges of (residual) capacity $< \Delta$.

↳ If $\exists$ a s-t path in $G_f(\Delta)$ then the flow can be augmented by Δ.

$\Rightarrow$ Augmentation of flow by Δ in the original graph.

$$|C| \leftarrow \min\left\{ \sum_{w} c(s \to w), \sum_{u} c(u \to t) \right\}$$

Start with Δ that is maximal power of 2 s.t $\Delta < |C|$.

Preprocessing: $\Delta = \max\{2^k \mid 2^k < |C|\}$.

$\frac{|C|}{2} \le \Delta \le |C|$

$\boxed{\Delta \le F \le 2\Delta}$.

→ Compute $s \rightsquigarrow t$ path and augment the flow until there is no longer a $s \rightsquigarrow t$ path in $G_{f'}(\Delta)$.

↳ 1. Find $s \rightsquigarrow t$ path in $G_f(\Delta)$ if it exists.

↳ if s-t path exists then update the flow. f'.

$|f'| \ge |f| + \Delta$.

$|f'| = |f| + F$

$\Delta \le F \le |C| - |f|$.

$\le 2m$

$\le \log_2 |C| + 1$

$\Delta \ge 1$.

while

Consider $G_{f'}(\Delta)$ and repeat step 1.

↳ Else, there is (S,T) cut in $G_{f^*}(\Delta)$.
Update Δ to $\frac{\Delta}{2}$ and construct $G_f(\Delta/2)$.
Go to step 1.

$$\Delta = \max\{2^k \mid 2^k < |C|\}.$$

if $\Delta = 2^{k^*}$ Then $2^{k^*+1} \geq |C|$. $\Rightarrow$ No. of reductions / iterations wrt Δ $\leq \lceil \log_2 |C| \rceil + 1$

First step.

Look for $s \rightsquigarrow t$ paths with bottleneck capacity of at least $\frac{|C|}{2}$ (if they exist).

$$0 \xrightarrow{\geq} \frac{C}{2} \xrightarrow{\geq} \frac{C}{4}$$
$\vdots$

This approach works better when

$$O(\log_2 |C| \cdot \underline{\underline{L}} \cdot (m+n)) \leq O(|C| \cdot (m+n)) . \checkmark$$

$\leq 2m$.

if $|C| = 2^n$

$O(n \cdot m \cdot (m+n)) \leq n^5$

$2^n \cdot (m+n) \sim n^2 \cdot 2^n$.

Lemma: In each Δ-phase, the no. of augmentations is at most $2m$.

→ Let $|f|$ be the flow at the end of the Δ-phase. Then the cut obtained in $G_f(\Delta)$ has capacity at most $|f| + m \cdot \Delta$. $\quad Cut_f(S,T)$

Max flow can at most be $|f| + m\Delta$.

$\Delta \to \frac{\Delta}{2}$. $\quad |f'| \geq |f| + L \cdot \frac{\Delta}{2}$. $\quad \frac{m\Delta}{2}$

Let $|f'|$ be the flow at the end of the $\frac{\Delta}{2}$-phase.

cut size $\leq |f'| + m \cdot \frac{\Delta}{2}$.

$|f'| \leq |Cut_f(S,T)|$
$\leq |f| + m\Delta$.

$L \cdot \frac{\Delta}{2} \leq |f'| - |f| \leq m\Delta$

$|f'| \leq |f| + m\Delta \Rightarrow \frac{|f'| - |f|}{\frac{\Delta}{2}} \leq \frac{m\Delta}{\frac{\Delta}{2}} = 2m$.

total Augmentation, in a $\frac{\Delta}{2}$-phase → $|f'| - |f| \geq L \cdot \frac{\Delta}{2}$

Obtaining f' from from f by doing $L, \geq \frac{\Delta}{2}$ augmentations.

$L \cdot \frac{\Delta}{2} \leq |f'| - |f| \leq m\Delta \Rightarrow L \leq 2m$.

Start:

$$\frac{|C|}{2} \leq \Delta \leq |C|$$

→ If there is an $s \leadsto t$ path P, then augment the flow with $\min_{u \to v \in P} \{c(u \to v)\}$.

$$\underbrace{0 + \min_{u \to v \in P} \{c(u \to v)\}}_{+\Delta} \geq \underbrace{\Delta}_{+\Delta} \geq \frac{|C|}{2}.$$

$$\geq + \frac{|C|}{2}$$

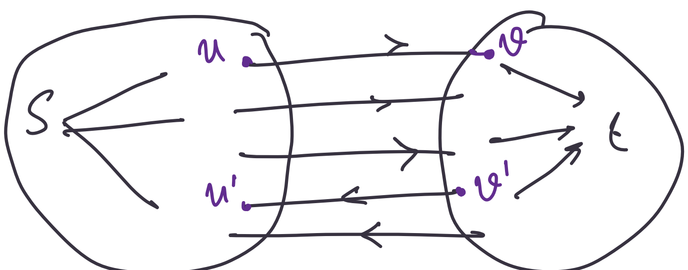
Lemma: Let $|f|$ be the flow at the end of the Δ-phase. Then the cut obtained in $G_f(\Delta)$ has capacity at most

$$|f| + m \cdot \Delta.$$

$$G_f(\Delta) \xrightarrow{f'} G_{f'}(\Delta).$$

$u \to v$ $\quad$ $\cancel{u \to v}$ only if $c_e - f'_e \geq \Delta$.

Pf:



Claim: $\forall e \in$ cut. $c_e < f_e + \Delta$ for forward edges and $f_e < \Delta$ for backward edges.

Otherwise $c_e \geq f_e + \Delta$ $\quad$ $u \to v$ $\quad$ $c_e - f_e$ $\quad$ $f_e \geq \Delta$

$\hookrightarrow$ $v \in S$ $\quad$ $\Rightarrow$ $v' \xrightarrow{\text{flow}} u'$

An edge $\in$ cut if it can no longer carry a flow of Δ.
in $G_f(\Delta)$

$|f|$ = Flow across = $\sum_{\substack{e \\ \text{fwd edges in cut}}} f_e - \sum_{\substack{e' \\ \text{back edges}}} f_{e'}$

$c_e < f_e + \Delta$

$\geq \sum_{\substack{e \\ \text{fwd edges}}} (c_e - \Delta) - \sum_{\substack{e' \\ \text{back ward edges.}}} \Delta$ $\quad f_{e'} < \Delta$.

$\geq \sum_{\substack{\text{fwd} \\ \text{edges}}} c_e - \sum_{e,e'} \Delta$

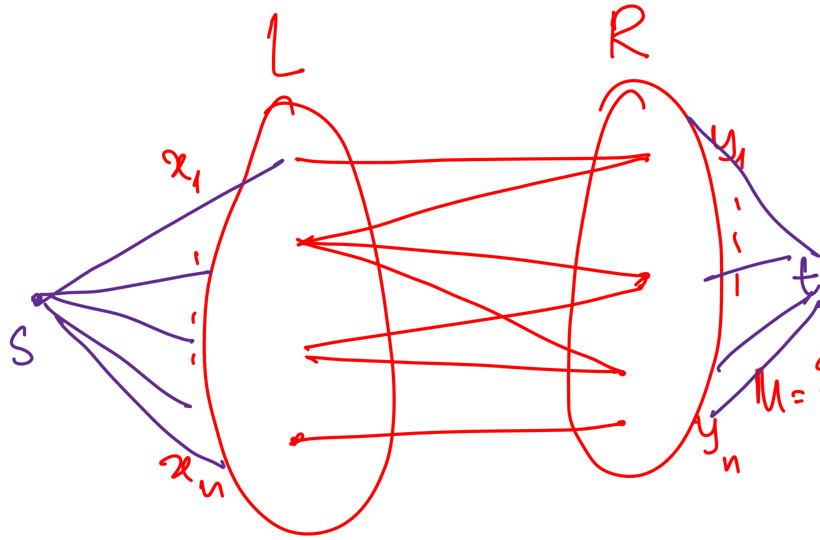
$\geq \sum c_e - m\Delta$

$\sum_{\substack{e \text{ fwd} \\ \text{edges}}} c_e \leq |f| + \underbrace{m\Delta}_{2m\Delta}$

Bipartite matching.

(V_1, V_2, E)

$G = (L, R, E)$



Perfect matching:

$M \subseteq E$ s.t.

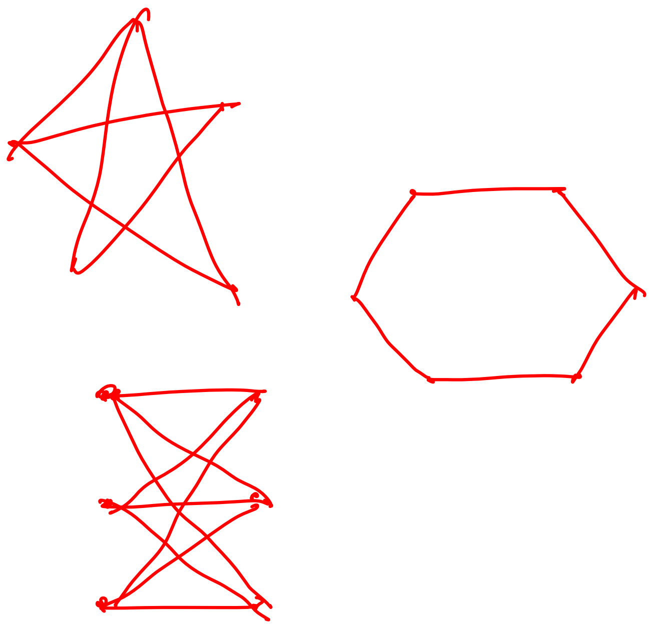
$M = \{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)\}$

$\downarrow$

x_i belongs to exactly one edge in M

and

y_i belongs to exactly one edge in M.



<u>Thm</u>: If $\exists$ a PM then using maxflow-mincut, we can obtain the PM.

$\rightarrow$ Bipartite matching reduces to finding a flow of value/size n in the updated graph.