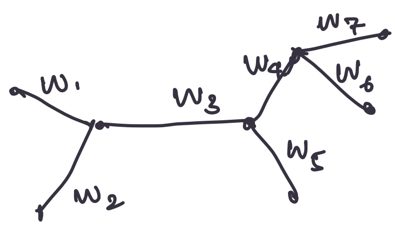
# Graph algorithms (contd).

## Minimum Spanning Tree.

Graph $G=(V,E)$: Spanning tree is a tree that covers all vertices.

Minimum spanning tree is the spanning tree w/ min wt/cost.

$T =$ 


$wt(T) = w_1 + w_2 + \cdots + w_7$.

Assumption: All edges get distinct wts.

$$\text{cost}(T) = \sum_{e \in T} \text{cost}(e).$$

Claim: Under this assumption, there is a unique MST.

Say $\underbrace{T_1 \text{ and } T_2}$ both attain min cost of $c^*$.

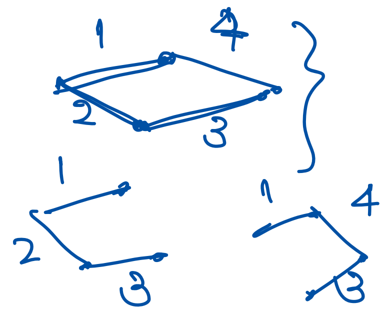
$\underline{E(T_1)}$ . $\underline{E(T_2)}$ are distinct subsets of $E$.

$\to$ span $V$

$E_1'$ $E_2'$ $V'$

$V' \subseteq V$

$V'' = V \setminus V'$



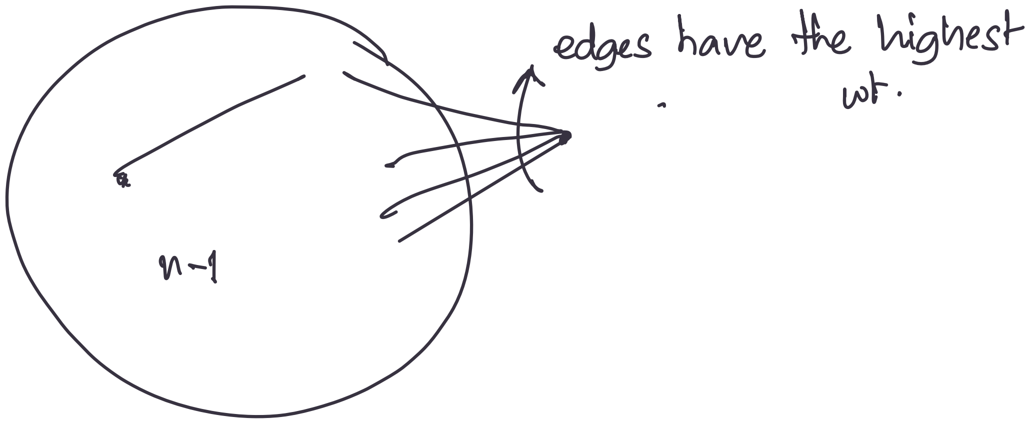

Swapping arguments.

$G = (V, E)$, costs for every edge.

- Prim's
- Kruskal's

- Pick edges of smallest wt/cost iteratively and maintain tree structure.

$e_1, e_2, \ldots, e_m$.

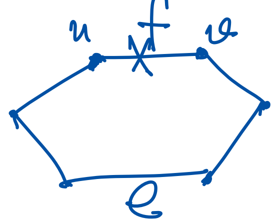

- Iteratively grow the set $S$ until $S = V$:

  Find an edge of min cost s.t it has one end vertex in $S$ and the other end in $V \setminus S$.

- Reverse-delete.

- Cycle property: Let $C$ be any cycle. Let edge $f$ be the max. wt edge in $C$. Then $f$ cannot be a part of MST.

Pf: Let $T^*$ be the MST s.t it contains $f$. $T^* - \{f\}$ makes it disconnected.



Choose another edge $e$ s.t $cost(e) < cost(f)$ and edge $e$ goes across the cut.

- In $T^* - \{f\}$, $u$ and $v$ are in distinct parts.

$\Rightarrow \exists$ an edge in $C$ that goes across the cut.

$T = T^* \cup \{e\} - \{f\}$.

$cost(e) < cost(f)$

$\Rightarrow cost(T) < cost(T^*)$.

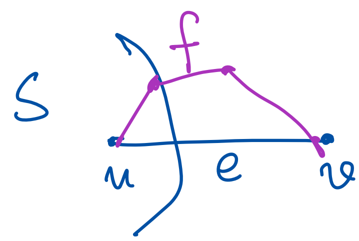
$\to\!\!\times\!\!\leftarrow$ Contradiction to $T^*$ being MST

$1 \leq |S| \leq n-1$

· Cut property: Let $S$ be a set of vertices. Let $e$ be the edge of min cost among the edges that have exactly one end point in $S$. Then MST contains $e$.

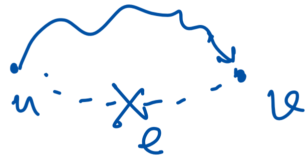
Pf: Assume that MST $T^*$ does not contain $e$.

$u \in S$ and $v \notin S$.




in $G$

$\exists$ edge $f$ s.t $f$ connects $u$ and $v$ and one end point of $f$ is in $S$ and the other $\notin S$.




in $T^*$

By the min cost argument, $cost(e) < cost(f)$.

$T = T^* \cup \{e\} - f$.

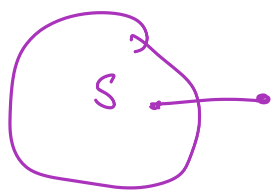
$cost(e) < cost(f)$

$\Rightarrow \underbrace{cost(T) < cost(T^*)}_{\text{Contradiction to our assumption.}}$

Kruskal's algorithm:

- Add edges of min possible wt s.t they donot lead to cycles. ← Cycle property + considering edges in incr. order of cost + cut property.

Correctness of Kruskal's is guaranteed by both cut and cycle properties.

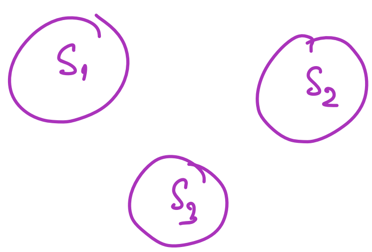



- Correctness of Prim's is guaranteed by cut property.

$n$ ExtractMin + Maintaing the tree. } Similar to Dijkstra's

Union ((u,v))
↑
Merge

(u, v)
✓ Find(u) $\stackrel{?}{=}$ Find(v)
↑ Outputs the index of the set containing u.




Kruskal's:

$n$ Union operations
$2m$ Find