

Slightly Improved Lower Bounds for Homogeneous Multi- r -ic Formulas of Bounded Depth

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Abstract

Kayal, Saha and Tavenas (Theory of Computing, 2018), showed that any bounded-depth homogeneous formula of bounded individual degree (bounded by r) that computes an explicit polynomial over n variables must have size $\exp\left(\Omega\left(\frac{1}{r}\left(\frac{n}{4}\right)^{1/\Delta}\right)\right)$ for all depths $\Delta \leq O\left(\frac{\log n}{\log r + \log \log n}\right)$. In this article we show an improved size lower bound of $\exp\left(\Omega\left(\frac{\Delta}{r}\left(\frac{nr}{2}\right)^{1/\Delta}\right)\right)$ for all depths $\Delta \leq O\left(\frac{\log n}{\log r}\right)$, and for the same explicit polynomial. In comparison to Kayal, Saha and Tavenas (Theory of Computing, 2018) (1) our results give superpolynomial lower bounds in a wider regime of depth Δ , and (2) for all $\Delta \in \left[\omega(1), o\left(\frac{\log n}{\log r}\right)\right]$ our lower bound is asymptotically better.

This improvement is due to a finer product decomposition of general homogeneous formulas of bounded-depth. This follows from an adaptation of a new *product decomposition* for bounded-depth multilinear formulas shown by Chillara, Limaye and Srinivasan (SIAM Journal of Computing, 2019).

1 Introduction

One of the major focal points in the area of algebraic complexity theory is to show that certain polynomials are hard to compute syntactically. Here, the hardness of computation is quantified by the number of arithmetic operations that are needed to compute the target polynomial. Instead of the standard Turing Machine model, we consider Arithmetic Circuits and Arithmetic Formulas as models of computation. We refer the readers to the standard resources [1, 2] for more information on arithmetic formulas and arithmetic circuits. Without loss of generality, in this paper we assume that our formulas are layered with alternate layers of $+$ -nodes and \times -nodes, with a single output node labeled by $+$ (sometimes suitably denoted symbolically as $\Sigma\Pi\Sigma\cdots\Sigma\Pi\cdots\Pi\Sigma$). Product-depth refers to the maximum number of \times -nodes encountered on any leaf-to-output path. Thus product-depth is related to depth by a factor of at most 2.

Valiant conjectured that Permanent does not have polynomial sized arithmetic circuits [3]. Thus the holy grail is to prove superpolynomial size lower bounds against arithmetic circuits and formulas. However, the best known circuit size lower bound is $\Omega(n \log n)$, for a Power Symmetric polynomial, due to Baur and Strassen [4, 5], and, the best known formula size lower bound is $\Omega(n^2)$, due to Kalorkoti [6]. Due to the slow progress towards proving general circuit/formula lower bounds, it is natural to study some restricted classes of arithmetic circuits and formulas.

Since most of the polynomials of interest like Determinant, Permanent, etc. are multilinear polynomials, it is natural to consider the restriction where every intermediate computation is in fact multilinear. Due to the phenomenal work in the last two decades [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17], the complexity of multilinear formulas and circuits is better understood than that of general formulas and circuits.

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Motivation for this work Hrubeš and Yehudayoff [12] showed that any homogeneous multilinear formula that computes an Elementary Symmetric polynomial¹ (of “large enough” degree) must be of quasipolynomial size. Further they showed that any homogeneous multilinear formula of bounded depth that computes an Elementary Symmetric polynomial (of “large enough” degree) must be of exponential size. They prove this by first exhibiting a structural property of homogeneous multilinear formulas (of bounded depth resp.) – they show that any polynomial that is computed by a homogeneous multilinear formula can be expressed as a sum of a very few summands each of which is a product of a large number of factors. We shall hence forth refer to such an expression as *Product Decomposition*. They then use the structure obtained from product decomposition to prove that, if the size of such formulas is strictly less than quasipolynomial (exponential resp.) then the polynomial computed is “sparse”. They show that Elementary Symmetric polynomial (of “large enough” degree) is not “sparse” and thus infer that it can not be computed by such formulas.

Kayal, Saha and Tavenas [18] have shown that even homogeneous Multi- r -ic Arithmetic Formulas (cf. Definition 2.2) can be expressed as a *small* sum of *large* products. Using this, they prove lower bounds against homogeneous multi- r -ic formulas using a suitable notion of *sparsity* which is the number of r -extremal monomials computed, as the complexity measure. In particular, they showed that for an explicit polynomial $P_{n,r}$ (cf. Definition 2.5) over n variables and degree $nr/2$, any homogeneous multi- r -ic formula of depth $\Delta \leq O\left(\frac{\log n}{\log r + \log \log n}\right)$ computing it must have size $\exp\left(\Omega\left(\frac{1}{r}\left(\frac{n}{4}\right)^{1/\Delta}\right)\right)$.

In this article, we first prove a refined product decomposition² (cf. Lemma 3.2) for general homogeneous formulas of bounded depth – we show that homogeneous formulas of bounded depth can be expressed as a *small* sum of *larger* products. This is analogous to the product decomposition in [15] for bounded depth multilinear formulas. This structure can be further transformed into a *required* product decomposition (to prove lower bounds) for homogeneous multi- r -ic formulas (cf. Lemma 4.2). That is, in comparison to [18] we show that each summand in the *required* product decomposition has more factors. As a result of this, we get quantitatively better lower bounds than [18] for homogeneous multi- r -ic formulas at bounded depths.

Theorem 1.1. *Let Δ, n and r be integers such that n is even, $\Delta \leq O(\log n / \log r)$ and $r^{(\Delta-1)/\Delta} = o(\Delta(n/2)^{1/\Delta})$. Then any homogeneous multi- r -ic formula of product-depth Δ computing $P_{n,r}$ must have size $2^{\Omega\left(\frac{\Delta}{r}\left(\frac{nr}{2}\right)^{1/\Delta}\right)}$.*

Our lower bound is superpolynomial for all depths which are $o\left(\frac{\log n}{\log r}\right)$ while [18] gives a superpolynomial lower bound for only those depths that are $o\left(\frac{\log n}{\log r + \log \log n}\right)$. It is important to remark that this lower bound is exponentially far from the current known upper bounds [20].

The following corollary easily follows from Theorem 1.1.

Corollary 1.2. *Let Δ, n and r be integers such that n is even, $\Delta \leq O(\log n / \log r)$ and $r^{(\Delta-1)/\Delta} = o(\Delta(n/2)^{1/\Delta})$. Then any homogeneous multi- r -ic circuit of product-depth Δ computing $P_{n,r}$ must be of size $2^{\Omega\left(\frac{1}{r}\left(\frac{nr}{2}\right)^{1/\Delta}\right)}$.*

¹The Elementary Symmetric polynomial of degree d on n variables, denoted by $\text{ESym}_n^d(x_1, x_2, \dots, x_n)$, is defined as $\sum_{\substack{S \subseteq [n] \\ |S|=d}} \prod_{i \in S} x_i$.

²This finer product decomposition for homogeneous formulas was observed independently by Herve Fournier, Guillaume Lagarde, Nutan Limaye and Srikanth Srinivasan [19]

2 Preliminaries

Notation:

- For a polynomial f , $\deg(f)$ denotes its total degree.
- For a monomial m , $\deg_{x_i}(m)$ denotes the degree of the variable x_i in m .
- For a polynomial f , $\deg_{x_i}(f)$ denotes the maximum over $\deg_{x_i}(m)$ over all the monomials m that appear in f . That is, $\deg_{x_i}(f) = \max_{m \in \text{monomials}(f)} \{ \deg_{x_i}(m) \}$.
- For a monomial m , $\text{Supp}(m)$ denotes the set of variables $\{x_i \mid \deg_{x_i}(m) \geq 1\}$.
- For a polynomial f , $\text{Supp}(f)$ denotes the set $\cup_{m \in \text{monomials}(f)} \text{Supp}(m)$.

Arithmetic Formulas:

Definition 2.1. An arithmetic formula Φ is said to be homogeneous if the following is satisfied for all nodes u in Φ .

- If $u \in \Phi$ is a sum gate, i.e., $u = u_1 + u_2 + \dots + u_t$ then $\deg(f_u) = \deg(f_{u_i})$ for all $i \in [t]$.
- If $u \in \Phi$ is a product gate, i.e., $u = u_1 u_2 \dots u_t$ then $\deg(f_u) = \sum_{i \in [t]} \deg(f_{u_i})$.
- If $u \in \Phi$ is a leaf, then $\deg(f_u)$ is equal to 1 or 0 depending upon whether the label is a variable or a constant respectively.

◇

Definition 2.2 (multi- \mathbf{r} -ic formulas). Let $\mathbf{r} = (r_1, r_2, \dots, r_n)$. An arithmetic formula Φ is said to be a syntactically multi- \mathbf{r} -ic formula if for all product gates $u \in \Phi$ and $u = u_1 \times u_2 \times \dots \times u_t$, each variable x_i can appear in at most r_i many u_i 's ($i \in [t]$), and the total degree with respect to every variable x_i over all the polynomials computed at u_1, u_2, \dots, u_t , is bounded by r_i , i.e. $\sum_{j \in [t]} \deg_{x_i}(f_{u_j}) \leq r_i$, for all $i \in [n]$. If $\mathbf{r} = (r, r, \dots, r)$, then we simply refer to them as multi- r -ic formulas.

◇

In this paper, we are only concerned about the homogeneous syntactically multi- r -ic formulas of bounded depth.

Definition 2.3 (Extremal monomials). Let $\mathbf{r} = (r_1, r_2, \dots, r_n)$. A monomial $m(\mathbf{x})$ is said to be \mathbf{r} -extremal if the degree of a variable x_i (for all i) in $m(\mathbf{x})$ is either 0 or r_i . If $r_1 = r_2 = \dots = r_n = r$, we simply call it r -extremal.

◇

Kayal, Saha and Tavenas [18] have generalized the complexity measure introduced by Hrubeš and Yehudayoff [12] for multilinear polynomials³.

Definition 2.4 (Complexity Measure: r -extremal monomials). The complexity measure that we consider for a polynomial $f(\mathbf{x})$ with respect to a parameter $r \geq 1$ is the total number of r -extremal monomials in f .

◇

Definition 2.5 (Hard polynomial). Let n, r be integers such that n is even and $r \geq 1$. The polynomial $P_{n,r}(\mathbf{x})$ over the variables $\mathbf{x} = \{x_1, x_2, \dots, x_n\}$ is defined as $P_{n,r}(\mathbf{x}) = \sum_{S \subseteq [n]} \prod_{i \in S} x_i^r$.

◇

³In spirit, the measure of Hrubeš and Yehudayoff [12] is similar to the complexity measure introduced by Jerrum and Snir [21] for monotone formulas.

It is easy to see that the polynomial $P_{n,r}$ can be expressed in terms of Elementary Symmetric polynomial of degree $n/2$ over n variables, i.e. $\text{ESym}_n^{n/2}(x_1^r, x_2^r, \dots, x_n^r)$.

We need the following proposition to help us obtain the “required” product decomposition (cf. Lemma 4.2) to help us obtain better lower bounds.

Proposition 2.6. *Let t and r be integers such that $r < t$. Let T be a syntactically multi- r -ic product of t polynomials $\{g_1, g_2, \dots, g_t\}$ over n variables $\{x_1, x_2, \dots, x_n\}$ such that $|\text{Supp}(g_i)| \geq 1$ for all $i \in [t]$. Then T can also be expressed as a syntactically multi- r -ic product of at least $t' > (t - r)/2$ many polynomials $\{h_1, h_2, \dots, h_{t'}\}$ over the same set of variables, such that $|\text{Supp}(h_i)| \geq 2$ for all $i \in [t']$.*

Proof. If $|\text{Supp}(g_i)| \geq 2$ for all $i \in [t]$, then we already have T in the required form. Else, split the factors $\{g_i \mid i \in [t]\}$ into $n + 1$ bags $B_0, B_1, B_2, \dots, B_n$ such that $B_0 = \{g_i \mid |\text{Supp}(g_i)| \geq 2\}$, and for all $i \in [n]$, $B_i = \{g_i \mid |\text{Supp}(g_i)| = 1; \text{Supp}(g_i) = \{x_i\}\}$. Some of these bags could potentially be empty. By suitable renaming, let S_1, S_2, \dots, S_n be the enumeration of the bags B_1, B_2, \dots, B_n in the non decreasing order of their cardinalities. Note that for all i , each element of the bag S_i is a univariate polynomial. Algorithm 1 describes the process of pairing two univariate polynomials to obtain polynomials of support 2. By invoking Algorithm 1 with $j = 1$, we finally get the set S_n , a set of univariate polynomials that could not be paired. Merge the leftover univariate polynomials in S_n with another factor of support at least 2.

Algorithm 1: Pair(j)

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1 if  $j=n$  then
2   | return  $S_n$ ;
3 else
4   | Pair each element of  $S_j$ , sequentially, with the first  $|S_j|$ -many elements of  $S_{j+1}$ ;
5   | Update  $S_{j+1}$  by removing all its elements that are already paired;
6   | return Pair( $j + 1$ );
7 end

```

Cardinality of S_n is at most r and thus using the aforementioned procedure, we have paired up at least $\sum_{i=1}^n |S_i| - r$ many polynomials to generate at least $\lfloor (\sum_{i=1}^n |S_i| - r)/2 \rfloor$ many new polynomials of support 2. Recall that $|B_0| + \sum_{i=1}^n |S_i| = t$. Post the procedure, the total number of polynomials that have support at least 2 is at least

$$|B_0| + \left\lfloor \frac{(\sum_{i=1}^n |S_i| - r)}{2} \right\rfloor \geq \left\lfloor \frac{|B_0| + (\sum_{i=1}^n |S_i| - r)}{2} \right\rfloor \geq \frac{t - r}{2} - 1.$$

□

3 Product Decomposition for Homogeneous Formulas of Bounded Depth

Before we derive a product decomposition for homogeneous formulas of bounded depth, we need the following definition.

Definition 3.1. *A homogeneous polynomial f is said to be a (t, L) -product if there exist t homogeneous polynomials $\{f_1, \dots, f_t\}$ such that*

1. $f = f_1 f_2 \dots f_t$,

2. $|\text{Supp}(f_i)| \geq L$ for all $i \in [t]$.

◇

Lemma 3.2. *Let d and Δ be integers such that $1 \leq \Delta < \log d$. Let f be a polynomial of total degree d over n variables that is computed by a homogeneous formula of size s and product-depth Δ . Then f can also be expressed as a sum of at most s -many $(v, 1)$ -products where $v \geq \Delta(d^{1/\Delta} - 1)$.*

Proof. We shall prove by induction on Δ that f can be expressed as a sum of at most s -many $(v, 1)$ -products where $v \geq \Delta(d^{1/\Delta} - 1)$.

Base Case: $\Delta = 1$ Let the polynomial f be computed by a homogeneous depth three $(\Sigma\Pi\Sigma)$ formula Φ of size s and degree d . Here $f = \sum_{i=1}^{s_0} T_i$ and $T_i = \prod_{j=1}^d f_{i,j}$. It is important to note that each summand T_i has exactly d factors and each $f_{i,j}$ is a homogeneous linear polynomial with $|\text{Supp}(f_{i,j})| \geq 1$. Since $f = \sum_{i=1}^{s_0} T_i$, f can now be expressed as a sum of at most $s_0 \leq s$ many $(d, 1)$ -products and trivially $d > (d - 1)$.

Induction Step Let f be computed by a $(\Sigma\Pi)^\Delta \Sigma$ formula Φ of size s . Let $\Phi_1, \dots, \Phi_{s_0}$ (for some $s_0 \leq s$) be the sub-formulas feeding into the output node in Φ , computing the polynomials f_1, f_2, \dots, f_{s_0} respectively. That is, $f = f_1 + \dots + f_{s_0}$. Since Φ is homogeneous, $\deg(f) = \deg(f_i)$ for all $i \in [s_0]$. Let $f_i = \prod_{j=1}^{k_i} f_{i,j}$ for all $i \in [s_0]$. Note that each sub-formula $\Phi_{i,j}$ computing $f_{i,j}$ is a $(\Sigma\Pi)^{\Delta-1} \Sigma$ formula.

Consider an arbitrary polynomial f_i from $\{f_1, f_2, \dots, f_{s_0}\}$. Recall that $f_i = \prod_{j=1}^{k_i} f_{i,j}$ for some parameter k_i . Since the product of $f_{i,j}$'s is homogeneous, by an averaging argument we get that there exists a factor, say $f_{i,j'}$, in f_i such that $\deg(f_{i,j'})$ is at least d/k_i . By suitable renaming, we can assume that $f_{i,1}$ to be that factor.

By applying the induction hypothesis on $f_{i,1}$, we get that $f_{i,1}$ can be expressed as a sum of at most $s' = \text{size}(\Phi_{i,1})$ many $(v', 1)$ -products where $v' \geq (\Delta - 1)((d/k_i)^{1/(\Delta-1)} - 1)$. Multiplying each of these summands in the expression obtained for $f_{i,1}$ through inductive hypothesis, with the rest of the factors in f_i , we get that there are $v = v' + k_i - 1$ many factors in each of the summands in f_i . We now have at most $\text{size}(\Phi_{i,1}) \leq \text{size}(\Phi_i)$ many summands.

$$v = v' + k_i - 1 \geq \min_{k_i \in [d]} \left\{ (\Delta - 1) \left((d/k_i)^{1/(\Delta-1)} - 1 \right) + k_i - 1 \right\} \geq \Delta \left(d^{1/\Delta} - 1 \right).$$

The last inequality follows from the fact that the term before the inequality attains a minima at $k_i = d^{1/\Delta}$. Thus, we can express every f_i as a sum of at most $\text{size}(\Phi_i)$ many $(v, 1)$ -products. Recall that $\text{size}(\Phi) = 1 + \sum_{i=1}^{s_0} \text{size}(\Phi_i) = s$ and $f = \sum_{i=1}^{s_0} f_i$. Using this, we get that f can be expressed as a sum of at most s many $(v, 1)$ -products with $v \geq \Delta(d^{1/\Delta} - 1)$. This completes the proof. □

4 Lower Bound for Homogeneous Multi- r -ic Formulas of Bounded-Depth

Definition 4.1. *A homogeneous polynomial f is said to be in (v, L) -form if there exist v homogeneous polynomials $\{f_1, \dots, f_v\}$ such that*

1. $f = f_1 f_2 \dots f_v$,

2. $\left| \text{Supp}(f_i) \setminus \left(\cup_{j < i} \text{Supp}(f_j) \right) \right| \geq L$ for all $i \in [v]$.

◇

Using [Lemma 3.2](#), we shall first show that homogeneous multi- r -ic formulas of bounded-depth can be expressed as a sum of a few summands each of which is in $(v, 2)$ -form where v is large enough.

Lemma 4.2. *Let r, d and Δ be integers such that $r = o(\Delta(d^{1/\Delta} - 1))$. Let f be a polynomial over n variables, of total degree d and individual degree r . Further, let f be computed by a homogeneous multi- r -ic formula of size s and product-depth Δ . Then f can also be expressed as a sum of at most s -many $(t, 2)$ -products where $t \geq \frac{(\Delta(d^{1/\Delta} - 1) - r)}{2} - 1$.*

Proof. From [Lemma 3.2](#), we get that f can be expressed as a sum of at most s -many $(v, 1)$ -products where $v \geq \Delta(d^{1/\Delta} - 1)$. Using [Proposition 2.6](#), we can transform each of these summands, which is a $(v, 1)$ -product, into a $(t, 2)$ -product where $t \geq \frac{v-r}{2} - 1$. Thus, we can express f as a sum of at most s -many $(t, 2)$ -products where $t \geq \frac{(\Delta(d^{1/\Delta} - 1) - r)}{2} - 1$. □

To prove the lower bound we need the following definition and two lemmas that were proved in [\[18\]](#) using elegant combinatorial arguments.

Lemma 4.3 (Lemma 5.5, [\[18\]](#)). *Let $t, r \geq 1$ be integers. If a multi- r -ic polynomial T is a $(t, 2)$ -product then T has a $(\frac{t}{2r} - 1, 2)$ -form.*

Lemma 4.4 (Bound on the extremal monomials, Lemma 4.6, [\[18\]](#)). *Let T be an n -variate polynomial in (v, L) -form. Then the number of r -extremal monomials in T is at most $\frac{2^n}{L^{v/2}}$.*

With this background, we can now prove an improved lower bound.

Proof of Theorem 1.1. Let Φ be a homogeneous multi- r -ic formula of product-depth Δ of size s computing $P_{n,r}$. Recall that $P_{n,r} = \sum_{S \subset [n]} \prod_{i \in S} x_i^r$ and $\deg(P_{n,r}) = \frac{nr}{2}$. From [Lemma 4.2](#) and [Lemma 4.3](#), we get that the computation of Φ can be expressed as a sum of at most s many terms each of which is in $(v, 2)$ -form where $v \geq \frac{(\Delta((nr/2)^{1/\Delta} - 1) - r - 2)}{4r} - 1$. By invoking the [Lemma 4.4](#), we get that the number of r -extremal monomials is at most $s \cdot \frac{2^n}{2^{v/2}}$.

It is easy to see that there are $\binom{n}{n/2}$ monomials in $P_{n,r}$ and each of these monomials is a r -extremal monomial.

Since Φ computes $P_{n,r}$, the number of r -extremal monomials of Φ must equal the number of r -extremal monomials in $P_{n,r}$. Thus by using the fact that $\binom{n}{n/2}$ is at most $2^n / \sqrt{\pi n}$ (from Stirling's approximation) we get the following.

$$s \cdot \frac{2^n}{2^{v/2}} \geq \binom{n}{n/2} \implies s \geq \frac{2^{v/2}}{2^n} \cdot \frac{2^n}{\sqrt{2n}} \geq \exp\left(\Omega\left(\frac{\Delta}{r} \cdot \left(\frac{nr}{2}\right)^{1/\Delta}\right)\right).$$

The floor in the expression for v can be subsumed into the Ω in the last inequality. This lower bound is superpolynomial when $\Delta \leq o\left(\frac{\log n}{\log r}\right)$. □

Proof of Corollary 1.2. Any circuit of product-depth Δ has depth at most $2\Delta + 1$. By replicating nodes, any homogeneous multi- r -ic circuit of size s can be converted into a homogeneous multi- r -ic formula of product-depth Δ and size $s^{2\Delta+1}$. From Theorem 1.1, we get that any homogeneous multi- r -ic formula of product-depth Δ computing $P_{n,r}$ must be of size at least $2^{\Omega\left(\frac{\Delta}{r}\left(\frac{nr}{2}\right)^{1/\Delta}\right)}$. Thus, $s^{2\Delta+1} \geq 2^{\Omega\left(\frac{\Delta}{r}\left(\frac{nr}{2}\right)^{1/\Delta}\right)}$ and $s \geq 2^{\Omega\left(\frac{1}{r}\left(\frac{nr}{2}\right)^{1/\Delta}\right)}$. This lower bound is strictly superpolynomial when $\Delta \leq o\left(\frac{\log n}{\log r + \log \log n}\right)$. \square

References

- [1] A. Shpilka, A. Yehudayoff, *Arithmetic circuits: A survey of recent results and open questions*, Foundations and Trends in Theoretical Computer Science 5 (2010) 207–388.
URL <http://dx.doi.org/10.1561/04000000039>
- [2] R. Saptharishi, *A survey of lower bounds in arithmetic circuit complexity*, github survey (2015).
URL <https://github.com/dasarpmar/lowerbounds-survey/releases/>
- [3] L. G. Valiant, *Completeness classes in algebra*, in: Proceedings Symposium on Theory of Computing (STOC), ACM, 1979, pp. 249–261. doi:10.1145/800135.804419.
URL <http://doi.acm.org/10.1145/800135.804419>
- [4] V. Strassen, Die berechnungskomplexität von elementarsymmetrischen funktionen und von interpolationskoeffizienten, Numerische Mathematik 20 (3) (1973) 238–251.
- [5] W. Baur, V. Strassen, The complexity of partial derivatives, Theoretical Computer Science 22 (1983) 317–330.
- [6] K. Kalorkoti, A lower bound for the formula size of rational functions, SIAM Journal on Computing 14 (3) (1985) 678–687.
- [7] N. Nisan, A. Wigderson, Lower bounds on arithmetic circuits via partial derivatives, Computational Complexity 6 (3) (1997) 217–234. doi:10.1007/BF01294256.
- [8] R. Raz, Separation of multilinear circuit and formula size, Theory of Computing 2 (1) (2006) 121–135. doi:10.4086/toc.2006.v002a006.
- [9] R. Raz, *Multilinear-NC² \neq multilinear-NC¹*, in: Proceedings of Foundations of Computer Science (FOCS), IEEE Computer Society, 2004, pp. 344–351. doi:10.1109/FOCS.2004.42.
URL <https://doi.org/10.1109/FOCS.2004.42>
- [10] R. Raz, A. Yehudayoff, Balancing syntactically multilinear arithmetic circuits, Computational Complexity 17 (4) (2008) 515–535. doi:10.1007/s00037-008-0254-0.
- [11] R. Raz, A. Yehudayoff, Lower bounds and separations for constant depth multilinear circuits, Computational Complexity 18 (2) (2009) 171–207. doi:10.1007/s00037-009-0270-8.
- [12] P. Hrubeš, A. Yehudayoff, Homogeneous formulas and symmetric polynomials, Computational Complexity 20 (3) (2011) 559–578.
- [13] R. Raz, A. Shpilka, A. Yehudayoff, A lower bound for the size of syntactically multilinear arithmetic circuits, SIAM Journal of Computing 38 (4) (2008) 1624–1647. doi:10.1137/070707932.

- [14] N. Alon, M. Kumar, B. L. Volk, Unbalancing sets and an almost quadratic lower bound for syntactically multilinear arithmetic circuits, in: Proceedings of Computational Complexity Conference (CCC), Vol. 102 of LIPIcs, Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2018, pp. 11:1–11:16.
- [15] S. Chillara, N. Limaye, S. Srinivasan, Small-depth multilinear formula lower bounds for iterated matrix multiplication with applications, SIAM Journal of Computing 48 (1) (2019) 70–92.
- [16] S. Chillara, C. Engels, N. Limaye, S. Srinivasan, A near-optimal depth-hierarchy theorem for small-depth multilinear circuits, in: Proceedings of Foundations of Computer Science (FOCS), IEEE Computer Society, 2018, pp. 934–945.
- [17] S. Chillara, N. Limaye, S. Srinivasan, A quadratic size-hierarchy theorem for small-depth multilinear formulas, in: Proceedings of International Colloquium on Automata, Languages, and Programming (ICALP), Vol. 107 of LIPIcs, Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2018, pp. 36:1–36:13.
- [18] N. Kayal, C. Saha, S. Tavenas, [On the size of homogeneous and of depth-four formulas with low individual degree](#), Theory of Computing 14 (16) (2018) 1–46. doi:10.4086/toc.2018.v014a016. URL <http://www.theoryofcomputing.org/articles/v014a016>
- [19] S. Srinivasan, personal communication (2019).
- [20] M. Kumar, R. M. de Oliveira, R. Saptharishi, Towards optimal depth reductions for syntactically multilinear circuits, Electronic Colloquium on Computational Complexity (ECCC) 26 (2019) 19.
- [21] M. Jerrum, M. Snir, [Some exact complexity results for straight-line computations over semirings](#), J. ACM 29 (3) (1982) 874–897. doi:10.1145/322326.322341. URL <http://doi.acm.org/10.1145/322326.322341>